Vol. 86 (2020) *REPORTS ON MATHEMATICAL PHYSICS* No. 3

UNIQUENESS OF GIBBS MEASURES FOR AN ISING MODEL WITH CONTINUOUS SPIN VALUES ON A CAYLEY TREE

F. H. HAYDAROV

National University of Uzbekistan, Uzbekistan, 100174, Tashkent, Almazar district, Universitet street, 4
(e-mail: haydarov_imc@mail.ru)

SH. A. AKHTAMALIYEV

Tashkent state pedagogical university, Uzbekistan, 100183, Tashkent, Chilanzar district, avenue Bunyodkor, 27. (e-mail: Shamshod2101@gmail.com)

M. A. NAZIROV

National University of Uzbekistan, Uzbekistan, 100174, Tashkent, Almazar district, Universitet street, 4.
(e-mail: madalixon@inbox.ru)

and

B. B. QARSHIYEV

Karshi Engineering Economic Institute, Uzbekistan, 180100, Kashkadarya, Karshi city, Mustakillik street, 225.
(e-mail: qarshiyevb@inbox.ru)

(*Received April 7, 2020 — Revised May 6, 2020*)

In this paper we consider an Ising model with nearest-neighbour interactions with spin space $[0, 1]$ on a Cayley tree. We present a sufficient condition under which the Ising model has a unique splitting Gibbs measure.

Mathematics Subject Classifications (2010). 82B05, 82B20 (primary); 60K35 (secondary).

Keywords: Cayley tree, Ising model, Hamiltonian, limiting Gibbs measures, uniqueness.

1. Introduction

The description of infinite-volume (or limiting) Gibbs measures for a given Hamiltonian plays an essential role in the theory of equilibrium statistical mechanics. Such measures, for a wide class of Hamiltonians, were established in the groundbreaking work of Dobrushin [4]. However, a complete analysis of the set of limiting Gibbs measures for a specific Hamiltonian is often a difficult problem (e.g. [1, 2, 17–19]).

An increasing attention to models with spin values in [0,1] on Cayley trees has been given for ten years. There are some works on Gibbs measures for models with nearest-neighbour interactions with the set of spin values $[0, 1]$. The main result devoted to such models is the following: splitting Gibbs measures on the Cayley tree of order k are described by solutions to a nonlinear integral equation. For $k = 1$ (when the Cayley tree becomes a one-dimensional lattice $\mathbb{Z}$) it is shown that the integral equation has a unique solution, implying that there is a unique Gibbs measure (confirming a series of well-known results; e.g. [3, 11].) For general k, a sufficient condition is found under which a periodic splitting Gibbs measure is unique. On the other hand, on the Cayley tree Γ_k of order $k = 2$, the existence of phase transitions is proven, see [5, 8, 10, 12–14]. We note that all of these papers are devoted to models with nearest-neighbour interactions.

In [9, 13] the splitting Gibbs measures for four competing interactions (external field, nearest neighbour, second neighbours and triples of neighbours) of models on Γ_2 are described. Also, it is proven that periodic Gibbs measure for the Hamiltonians with four competing interactions is either *translation-invariant* or *periodic with period two*.

In [7] there is the following open problem: the number of translation-invariant splitting Gibbs measures for the Ising model with nearest-neighbour interactions with spin space $[0, 1]$ on Γ_2 is unknown. In this paper we study this open problem and get the following results: the uniqueness of translation-invariant splitting Gibbs measures for the anti-ferromagnetic Ising model on Γ_2 and if the temperature is greater than or equal to $\frac{1}{2J}\ln\frac{\sqrt{5}+1}{2}$ then there is a unique translation-invariant splitting Gibbs measure for the ferromagnetic Ising model on Γ_2, where $J \in \mathbb{R}\setminus\{0\}$ is the interaction term between neighbouring spins. Also, a sufficient condition of uniqueness for the fixed points of Hammerstein operator given in [5], is investigated and we obtain better estimations for the sufficient condition of uniqueness.

2. Preliminaries

A Cayley tree $\Gamma_k = (V, L)$ of order $k \geq 1$ is an infinite homogeneous tree, i.e. a graph without cycles, with exactly $k + 1$ edges incident to each of vertices. Here V is the set of vertices and L that of edges (arcs). Two vertices x and y are called nearest neighbours if there exists an edge $l \in L$ connecting them, which is denoted by $l = \langle x, y\rangle$.

Let Λ be a subset of V. A configuration on Λ is an arbitrary function $\sigma_\Lambda : \Lambda \to [0, 1]$, with values $\sigma(x)$, $x \in \Lambda$. The set of all configurations on $\Lambda \subset V$ is denoted by $\Omega_\Lambda = [0, 1]^\Lambda$ and $\Omega := \Omega_V$. Let $\bar{\sigma}_\Lambda$ be any fixed configuration on Λ, i.e. $\bar{\sigma}_\Lambda \in \Omega_\Lambda$. Then the following family of configurations

$$\{\sigma \in \Omega : \sigma|_\Lambda = \bar{\sigma}_\Lambda, \quad \Lambda \subset V\} \tag{2.1}$$

is called a cylinder with base $\bar{\sigma}_\Lambda$, where $\sigma|_\Lambda$ stands for the restriction of configuration $\sigma \in \Omega$ to Λ. If Λ is a finite set then (2.1) is called finite cylinder with base $\bar{\sigma}_\Lambda$.

Let $\mathcal{A}$ be the standard σ-algebra generated by finite cylinders. Now, we consider the (formal) Hamiltonian of Ising model with nearest-neighbour interactions as

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y), \tag{2.2}$$

where $J \in \mathbb{R} \setminus \{0\}$ is a coupling constant and $\langle x, y \rangle$ stands for nearest neighbour vertices and $\sigma \in \Omega$.

Note that if $J > 0$ then (2.2) gives rise to the ferromagnetic Ising model and if $J < 0$ then (2.2) gives rise to the anti-ferromagnetic Ising model.

The distance $d(x, y)$, $x, y \in V$, on Cayley trees is the length of (i.e. the number of edges in) the shortest path connecting x with y.

W_r stands for a 'sphere' and V_r for a 'ball' on the tree, of radius $r = 1, 2, \dots,$ centered at a fixed vertex x^0 (a root),

$$W_r = \{x \in V : d(x, x^0) = r\}, \quad V_r = \{x \in V : d(x, x^0) \leq r\}.$$

Denote

$$L_r = \{l = \langle x, y \rangle \in L : \ x, y \in V_r\}.$$

A probability measure μ on $(\Omega, \mathcal{A})$ is called a Gibbs measure (with the Hamiltonian H) if it satisfies the Dobrushin–Lanford–Ruelle (DLR) equation (see [4, 16]), namely for any $n = 1, 2, \dots$ and $\sigma_n \in \Omega_{V_n}$,

$$\mu\left(\left\{\sigma \in \Omega : \ \sigma\big|_{V_n} = \sigma_n\right\}\right) = \int_{\Omega} \mu(\mathrm{d}\omega) \nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n),$$

where $\nu^{V_n}_{\omega|_{W_{n+1}}}$ is the conditional Gibbs density depending on the inverse temperature $\beta = 1/T, \ T > 0$,

$$\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) = \frac{1}{Z_n(\omega)} \exp\left(-\beta H\left(\sigma_n, \ \omega\big|_{W_{n+1}}\right)\right).$$

Here and below, $\sigma_n : \ x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and $\omega \in \Omega_{W_{n+1}}$ (corresponding to σ_n). Also, $H\left(\sigma_n, \ \omega\big|_{W_{n+1}}\right)$ is defined as the sum $H(\sigma_n) + U\left(\sigma_n, \omega\big|_{W_{n+1}}\right)$, where

$$H(\sigma_n) = -J \sum_{\langle x, y \rangle \in L_n} \sigma_n(x)\sigma_n(y),$$

$$U\left(\sigma_n, \omega\big|_{W_{n+1}}\right) = -J \sum_{\langle x, y \rangle : \ x \in V_n, y \in W_{n+1}} \sigma_n(x)\omega(y).$$

Finally, $Z_n(\omega)$ stands for the partition function in V_n, with the boundary condition $\omega\big|_{W_{n+1}}$,

$$Z_n(\omega) = \int_{\Omega_{V_n}} \exp\left(-\beta H\left(\widetilde{\sigma}_n, \ \omega\big|_{W_{n+1}}\right)\right) \lambda_{V_n}(d\widetilde{\sigma}_n).$$

Here and below, λ is the Lebesgue measure on [0,1] (and can be considered as probability measure). Let $\Lambda \subset V$ be a finite set of cardinality $|\Lambda|$, then the set of all configurations on Λ is equipped with an a priori measure λ_Λ introduced as the $|\Lambda|$-fold power of λ.

REMARK 1. Note that $Z_n(\omega)$ is finite, since λ is a probability measure and

$$\widetilde{\sigma}_n \mapsto \exp\left(-\beta H\left(\widetilde{\sigma}_n, \omega|_{W_{n+1}}\right)\right)$$

is bounded on Ω_{V_n}.

Due to the nearest-neighbour character of the interaction, the Gibbs measure possesses a natural Markov property: for given a configuration ω_{n+1} on W_{n+1}, random configurations in V_n (i.e. 'inside' W_{n+1}) and in $V \setminus V_{n+1}$ (i.e. 'outside' W_{n+1}) are conditionally independent.

3. Main results

In this section we present a sufficient condition under which the Ising model has a unique splitting Gibbs measure. This condition is much better than the sufficient conditions of uniqueness of splitting Gibbs measures for the Ising model in [5, 9].

We use a standard definition of a translation-invariant measure (e.g. [17]). Let $h : [0,1] \times V \setminus \{x^0\} \to \mathbb{R}$ and $|h(t,x)| = |h_{t,x}| < C$, where x^0 is a root of the Cayley tree and C is a finite constant which does not depend on t. For some $n \in \mathbb{N}$ and $\sigma_n : x \in V_n \mapsto \sigma(x)$ we consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\Big(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\Big). \tag{3.1}$$

Here Z_n is the corresponding partition function,

$$Z_n = \int_{\Omega_{V_n}} \exp\Big(-\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_{\tilde{\sigma}(x),x}\Big)\lambda_{V_n}(d\tilde{\sigma}_n). \tag{3.2}$$

From the above, Z_n is the finite partition function.

A family of probability distributions $\mu^{(n)}$ is called compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$ it satisfies the condition

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)\lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \tag{3.3}$$

Here $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n. By the Kolmogorov extension theorem (see [15]), there exists a unique measure μ on Ω_V such that, for any $n \in \mathbb{N}$ and $\sigma_n \in \Omega_{V_n}$, $\mu\Big(\Big\{\sigma\Big|_{V_n} = \sigma_n\Big\}\Big) = \mu^{(n)}(\sigma_n)$.

The measure μ is called the *splitting Gibbs measure* corresponding to the Hamiltonian (2.2) and the function $x \mapsto h_{t,x}$, $x \neq x^0$.

Write $x < y$ if the shortest path from x^0 to y goes through x. Call vertex y a direct successor of x if $y > x$ and x, y are nearest neighbours. Denote by $S(x)$ the set of direct successors of x. Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

The following statement describes conditions on $h_{t,x}$, $x \neq x^0$, guaranteeing compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

PROPOSITION 1. *[12] The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots,$ in* (3.1) *are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds,*

$$f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(\theta t u) f(u, y) du}{\int_0^1 f(u, y) du}. \tag{3.4}$$

Here and below, $f(t, x) = \exp(h_{t,x} - h_{0,x})$, $t \in [0, 1]$ and $\theta = J\beta \in \mathbb{R} \setminus \{0\}$.

Note that $\mu^{(n)}(\sigma_n)$ depends on the model H, σ_n and β. In turn, because of H depends on J Eq. (3.4) depends on the parameter θ. Also, from Proposition 1 it follows that for any $h : [0, 1] \times V \setminus \{x^0\} \to \mathbb{R}$ satisfying (3.4) there exists a unique splitting Gibbs measure μ and vice versa.

The analysis of solutions to (3.4) is not easy. Therefore, we consider solutions in the class of translation-invariant functions $f(t, x)$, i.e. $f(t, x) = f(t)$, for any $x \in V$. For such functions and $k \in \mathbb{N}$, equation (3.4) can be written as

$$f(t) = \left(\frac{\int_0^1 e^{\theta t u} f(u) du}{\int_0^1 f(u) du} \right)^k. \tag{3.5}$$

Denote

$$(A_k f)(t) = \left(\frac{\int_0^1 e^{\theta t u} f(u) du}{\int_0^1 f(u) du} \right)^k, \qquad k \in \mathbb{N}. \tag{3.6}$$

For the case $k = 1$, the operator A_k has exactly one positive fixed point (see [12]). That is why we consider the case $k \geq 2$. Denote

$$\mathcal{P}_k = \left\{ f \in C[0, 1] : 1 \leq f(t) \leq e^{\theta k} \right\}, \qquad k \geq 2.$$

Note that $\mathcal{P}_k$ is a closed and convex subset of $C[0, 1]$. It is easy to check that if $f \in C[0, 1]$ is a positive solution of the equation $A_k f = f$, then $f \in \mathcal{P}_k$. By virtue of article [5], the set $A_k(\mathcal{P}_k)$ is relatively compact in $C[0, 1]$. Thus, from Schauder's fixed point theorem one gets the following result.

PROPOSITION 2 ([5]). *The operator A_k has at least one positive fixed point in $\mathcal{P}_k$.*

For every $k \in \mathbb{N}$ we consider a specific type of Hammerstein integral operator H_k acting in $C[0, 1]$ as follows

$$(H_k f)(t) = \int_0^1 e^{\theta t u} f^k(u) du. \tag{3.7}$$

PROPOSITION 3 ([5]). *The operator $A_k f = f$ has a positive fixed point if and only if H_k has a positive fixed point in $C[0, 1]$.*

Put

$$\max_{t \in [0,1]} f(t) = f_{\max}, \qquad \min_{t \in [0,1]} f(t) = f_{\min}.$$

Now, we give a sufficient condition of uniqueness for the positive fixed point of A_k. We introduce the usual norm of $f \in C[0,1]$ defined by $\|f\| = \max_{t\in[0,1]} |f(t)| = |f|_{\max}$.

LEMMA 1. *Assume that the function $f \in C[0,1]$ changes its sign on* $[0,1]$. *Then for every $c \in \mathbb{R}$ the following inequality holds*

$$2\|f-c\| - \|f\| \geq |f_{\min}|\,. \tag{3.8}$$

Proof: By the conditions of Lemmas, there exist $t_1, t_2 \in [0,1]$ such that

$$f_{\min} = f(t_1) < 0, \qquad f_{\max} = f(t_2) > 0.$$

For the case $c = 0$, the proof of the lemma is trivial. We consider the case $c > 0$.

1. Let $|f_{\min}| \geq f_{\max}$, then $\|f\| = |f_{\min}| = |f(t_1)|$. Clearly,

$$2\|f-c\| = 2\max\{|f(t_1)-c|, |f(t_2)-c|\} = 2|f(t_1)-c|.$$

From the last equality, one gets

$$2\|f-c\| - \|f\| > 2|f(t_1)| - \|f\|.$$

Since $\|f\| = |f_{\min}|$, we obtain

$$2\|f-c\| - \|f\| \geq \|f\| = |f_{\min}|.$$

2. Let $|f_{\min}| < f_{\max}$. At first we check the case: $\|f\| \geq c$. Then

$$\|f\| = f_{\max} = f(t_2).$$

We have

$$2\|f-c\| = 2\max\{|f(t_1)-c|, |f(t_2)-c|\} = 2\max\{|f(t_1)|+c, f(t_2)-c\}.$$

From

$$2\max\{|f(t_1)|+c, f(t_2)-c\} \geq |f(t_1)| + f(t_2),$$

we obtain

$$2\|f-c\| - \|f\| \geq |f(t_1)| + f(t_2) - \|f\| = |f_{\min}|.$$

Now, let us check the case $\|f\| < c$, i.e. $\|f\| = f(t_2)$. Then

$$2\|f-c\| = 2\max\{|f(t_1)-c|, |f(t_2)-c|\}.$$

Namely,

$$2\|f-c\| = 2\max\{|f(t_1)|+c, c-f(t_2)\}.$$

Consequently,

$$2\|f-c\| - \|f\| \geq 2c + |f(t_1)| - f(t_2) - \|f\| = 2(c-f(t_2)) + |f(t_1)| \geq |f_{\min}|.$$

Thus, for the case $c \geq 0$ the proof of lemmas has been completed. If $c < 0$ then $f(t) - c$ can be written as $c_1 - g(t)$, where $g(t) = -f(t)$ and $c_1 = -c > 0$. Consequently, the inequality (3.8) is equivalent to

$$2\|g-c_1\| - \|g\| \geq |g_{\min}|\,.$$

This completes the proof. □

THEOREM 1. *Let* $\theta_{cr} = \frac{1}{2}\ln\frac{\sqrt{5}+1}{2}$. *For* $\theta \in (-\infty, \theta_{cr}]$, *the Ising model* (2.2) *has a unique translation-invariant splitting Gibbs measure on the Cayley tree of order two.*

Proof: By Proposition 1, to prove the uniqueness of translation-invariant Gibbs measures for the Ising model (2.2) on the Cayley tree of order two is equivalent to showing that there exists a unique translation-invariant solution of Eq. (3.4). In turn, from Proposition 3, finding positive solutions to this equation is equivalent to finding positive fixed points of the operator H_2. That is why it is sufficient to show that if θ belongs to $(-\infty, \theta_{cr}]$ the operator H_2 has exactly one positive fixed point. Since A_2 has at least one positive fixed point in $\mathcal{P}_2$ and Proposition 3, we can conclude that H_2 has at least one positive fixed point.

Now, we show that H_2 has exactly one positive fixed point. Assume that the operator H_2 has two distinct positive fixed points f_1 and f_2. Let $h(t) = f_1(t) - f_2(t)$, then we prove that $h(t)$ changes its sign on $[0, 1]$. Put

$$\delta_s := \delta_{\sup}(f_1, f_2) = \sup\{\delta \in [0, \infty) : f_1(t) - \delta f_2(t) > 0, \qquad \text{for all } t \in [0, 1]\}.$$

Then

$$f_1(t) - \delta_s f_2(t) = H_2(f_1)(t) - \delta_s H_2(f_2)(t) = \int_0^1 e^{\theta t u}\left(f_1^2(u) - \delta_s f_2^2(u)\right) du.$$

Thus,

$$f_1(t) - \delta_s f_2(t) = \int_0^1 e^{\theta t u}\left(f_1(u) - \sqrt{\delta_s} f_2(u)\right)\left(f_1(u) + \sqrt{\delta_s} f_2(u)\right) du. \qquad (3.9)$$

Suppose that $\delta_s \geq 1$, then since $f_1(t) \neq f_2(t)$ for some t, we get

$$f_1(u) - \sqrt{\delta_s} f_2(u) \geq 0 \text{ for all } u \in [0, 1] \text{ and } \int_0^1 (f_1(u) - \sqrt{\delta_s} f_2(u)) du > 0.$$

Indeed, if

$$\int_0^1 (f_1(u) - \sqrt{\delta_s} f_2(u)) du = 0$$

then, by definition of δ_s, one gets $f_1(u) = \sqrt{\delta_s} f_2(u)$ for all $u \in [0, 1]$. The last equality contradicts to f_1 and f_2 being two distinct positive fixed points. Hence, we obtain

$$f_1(t) - \delta_s f_2(t) = \int_0^1 e^{\theta t u}\left(f_1(u) - \sqrt{\delta_s} f_2(u)\right)\left(f_1(u) + \sqrt{\delta_s} f_2(u)\right) du > 0. \quad (3.10)$$

On the other hand, by definition of δ_s, there is $t_0 \in [0, 1]$ such that $f_1(t_0) - \delta_s f_2(t_0) = 0$. But, Eq. (3.9) contradicts the inequality (3.10). Hence, $\delta_s < 1$, i.e. $h(t)$ changes its sign on $[0, 1]$. We can say that the maximum value of $h(t) = f_1(t) - f_2(t)$ ($h_{\max}$), without loss of generality, is less than or equal to the absolute value of $h_{\min}$, i.e. $\|h\| \leq |h_{\min}|$ (otherwise, we choose $-h(t) = f_2(t) - f_1(t)$). As a result, by Lemma 1, one gets the following inequality,

$$2\|h-c\| - \|h\| \geq |h_{\min}| \geq \|h\| \Rightarrow \|h-c\| \geq \|h\|, \qquad c \in \mathbb{R}.$$

Let $c = (e^{2\theta} + e^{-2\theta}) \int_0^1 h(u)du$, then

$$\left\| h(t) - (e^{2\theta} + e^{-2\theta}) \int_0^1 h(u)du \right\| \geq \|h\|. \tag{3.11}$$

On the other hand,

$$h(t) = \int_0^1 e^{\theta tu}(f_1^2(u) - f_2^2(u))du.$$

By Cauchy's mean value theorem, we get

$$h(t) = \int_0^1 2e^{\theta tu}\xi(u)h(u)du, \tag{3.12}$$

where

$$\min\{f_1(t), f_2(t)\} \leq \xi(t) \leq \max\{f_1(t), f_2(t)\}, \qquad t \in [0,1]. \tag{3.13}$$

Let the image (range) of ξ be denoted by $\mathrm{Im}(\xi)$. Now, we show that $\mathrm{Im}(\xi) \subset [e^{-2\theta}, e^{\theta}]$. If $g \in H_2(C[0,1])$, then the following inequality holds: $g_{\min} \geq e^{-\theta} \cdot \|g\|$. Indeed, there exists a continuous function g_1 such that $g = H_2 g_1$. Then

$$g_{\min} \geq e^{-\theta} \cdot \int_0^1 \left(e^{\theta \cdot u}\right) g_1^2(u)du = e^{-\theta} \cdot \|g\|,$$

i.e.

$$g \in \mathcal{B} := \{f \in C[0,1] : \; f_{\min} \geq e^{-\theta} \cdot \|f\|\}.$$

From (3.13), it is sufficient to prove that any fixed point of H_2 belongs to the set $[e^{-2\theta}, e^{\theta}]$. Let f be a fixed point of H_2, then we have $\|f\| \leq e^{\theta}\|f\|^2 \Rightarrow e^{-\theta} \leq \|f\|$. Since $f \in \mathcal{B}$, one gets

$$f(t) \geq f_{\min} \geq e^{-\theta}\|f\| \geq e^{-2\theta}.$$

On the other hand, we estimate $f(t)$ from above, i.e.

$$f(t) = (H_2 f)(t) \geq \int_0^1 f^2(u)du \geq f_{\min}^2 \;\Rightarrow f_{\min} \leq 1.$$

From $f \in \mathcal{B}$ we obtain

$$f(t) \leq f_{\max} \leq e^{\theta} \cdot f_{\min} \leq e^{\theta}.$$

Hence

$$\mathrm{Im}(f) \subset [e^{-2\theta}, e^{\theta}] \Rightarrow \mathrm{Im}(\xi) \subset [e^{-2\theta}, e^{\theta}].$$

Consequently, for all $t, u \in [0,1]$ we have $e^{\theta tu}\xi(u) \in [e^{-2\theta}, e^{2\theta}]$. Thus, the following inequality holds,

$$\left|2e^{\theta tu}\xi(u) - (e^{-2\theta} + e^{2\theta})\right| \leq e^{2\theta} - e^{-2\theta}.$$

We multiply both sides by $|h(u)|$,

$$\left|2e^{\theta tu}\xi(u)h(u) - (e^{-2\theta} + e^{2\theta})h(u)\right| \leq (e^{2\theta} - e^{-2\theta})|h(u)|.$$

After integrating both sides of the last inequality, we have

$$\left|h(t) - (e^{-2\theta} + e^{2\theta})\int_0^1 h(u)du\right| < (e^{2\theta} - e^{-2\theta})\|h\|.$$

From (3.11), we get the inequality

$$\|h\| \leq \left\|h(t) - (e^{-2\theta} + e^{2\theta})\int_0^1 h(u)du\right\| < (e^{2\theta} - e^{-2\theta})\|h\|.$$

If θ satisfies the condition $e^{2\theta} - e^{-2\theta} \leq 1$ then the operator H_2 has exactly one fixed point. The last inequality is equivalent to the condition $\theta \in (-\infty, \theta_{\text{cr}}]$. □

From the above, it is clear that $\theta = J\beta$ and $\beta = 1/T$, where $T > 0$ is the temperature. If $\theta < 0$ then $J < 0$ and if $\theta > 0$ then $J > 0$. Taking into account these factors, one gets the following:

COROLLARY 1. *For the Ising model with spin values in [0,1] on the Cayley tree of order two the following statements are true:*

(1) *If the temperature T satisfies the condition $T \geq (1/2J)\ln(\sqrt{5} + 1/2)$ then there is a unique translation invariant splitting Gibbs measure for the ferromagnetic Ising model.*
(2) *There is a unique translation invariant splitting Gibbs measure for the anti-ferromagnetic Ising model.*

Let us present the following open problem in [7].

Open problem. The number of translation invariant Gibbs measures for the Ising model (2.2) on Γ_2 is unknown.

However, we give the sufficient condition of uniqueness of translation invariant Gibbs measures for the Ising model, for any $\theta > \theta_{cr}$ finding the number of translation invariant Gibbs measures for the ferromagnetic Ising model is still open.

Acknowledgements

The authors thank the referee for careful reading of the manuscript; in particular, for a number of suggestions which have improved the paper.

REFERENCES

[1] P. M. Bleher and N. N. Ganikhodjaev: On pure phases of the Ising model on the Bethe lattice, *Theor. Probab. Appl.* **35** (1990).
[2] P. M. Bleher, J. Ruiz and V. A. Zagrebnov: On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice, *J. Stat. Phys.* **79** (1995).

[3] Yu. R. Dashjan and Yu. M. Suhov: On the problem of the Gibbs description of random processes with the discrete time, *Soviet Math. Doklady-Doklady AN SSR 242* **3** (1978).

[4] R. L. Dobrushin: Gibbsian random fields for lattice systems with pairwise interactions, *Func. Anal. Appl.* **2(4)** (1969).

[5] Yu. Kh. Eshkabilov, F. H. Haydarov and U. A. Rozikov: Uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley tree, *Math. Phys. Anal. Geom.* **16(1)** (2013).

[6] Yu. Kh. Eshkabilov, F. H. Haydarov and U. A. Rozikov: Non-uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley Tree, *Jour. Stat. Phys.* **147** (2012).

[7] Yu. Kh. Eshkabilov, Sh. D. Nodirov and F. H. Haydarov: Positive fixed points of quadratic operators and Gibbs measures, *Positivity* **20(4)** (2016).

[8] F. H. Haydarov: New normal subgroups for the group representation of the Cayley tree, *Lobach. J. Math.* **39(2)** (2018).

[9] F. H. Haydarov: Fixed points of Lyapunov integral operators and Gibbs measures, *Positivity* **22(4)** (2018).

[10] R. Kotecky and S. B. Shlosman: First-order phase transition in large entropy lattice models, *Commun. Math. Phys.* **83** (1982).

[11] S. A. Pigorov and Ya. G. Sinai: *Theor. Math. Phys.* **25** (1975), Phase diagrams of classical lattice systems (Russian).

[12] U. A. Rozikov, Yu. Kh. Eshkabilov: *Math. Phys. Anal. Geom.* **13** (2010), On models with uncountable set of spin values on a Cayley tree: Integral equations.

[13] U. A. Rozikov and F. H. Haydarov: Four competing interactions for models with an uncountable set of spin values on a Cayley tree, *Theor. Math. Phys.* **191(2)** (2017).

[14] U. A. Rozikov and F. H. Haydarov: *Inf. Dim. Anal. Quan. Prob.* **18** (2015), Periodic Gibbs measures for models with uncountable set of spin values on a Cayley tree.

[15] B. A. Robert and A. D. Catherine: *Probability and Measure Theory*, Academic Press 1999.

[16] O. E. Lanford and D. Ruelle: Observables at infinity and states with short range correlations in statistical mechanics, *Commun. Math. Phys.* **13(3)** (1969).

[17] Ya. G. Sinai: *Theory of Phase Transitions: Rigorous Results*, Pergamon, Oxford 1982.

[18] F. Spitzer: Markov random fields on an infinite tree, *Ann. Prob.* **3** (1975).

[19] S. Zachary: Countable state space Markov random fields and Markov chains on trees, *Ann. Prob.* **11** (1983).